PMM U.S.S.R.,Vol.49,No.3,pp. 367-374,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

ON SOME PROPERTIES OF OPTIMAL THERMOELASTIC DESIGNS IN THE CASE OF FIXED STRESS AND DEFORMATION FIELDS*

P.A. KUNTASHEV and YU.V. NEMIROVSKII

Sets of distributions of elastic moduli sare studied, giving rise to identical deformation or stress fields in a thermoelastic body. The affine character of the sets is proved and their interdependence is studied. It is shown that the problem of the distribution of the elastic parameters realizing a minimal stress level for the given deformation field represents a problem of convex programming. The case of the optimal design of a thermoelastic beam is discussed.

In view of the wide range of possibilities of controlling new technological methods of creating new materials and structures, the problem arises of optimal design, i.e. of constructing such fields of elastic parameters, which would ensure the stress-strain field most suitable for use. The problem of the optimal distribution of the Lamé parameters in an elastic body was studied in /1/ using the condition of least work done by the external forces. The distribution of the Lamé parameters maximizing the torsional rigidity of a prismatic rod was studied in /2/. The problem of the optimal distribution of Young's modulus in a rod was solved in /3/ using the condition of maximum critical load causing loss of stability. The distribution of Young's modulus in a prestressed beam was obtained in /4/ from the requirement that the highest first eigenfrequency be realized.

In many cases it is important not only to improve the rigidity characteristics of the constructions, but also to reduce the stress level. Below the authors investigate the general properties of the sets of elastic moduli distributions ensuring, for fixed loads and the temperature field, the realization of one and the same deformation or stress field. It is established that the sets have an affine structure, intersect at a unique point, and, that the tangent spaces of these sets are mutually complementary.

1. Let us consider a inhomogeneous, linearly elastic isotropic body occupying a finite volume $V \subset R^3$ with piecewise smooth surface A, at the point A_1 of which the displacements u_i are equal to zero. The remaining part A_2 is acted upon by a surface load p_i, which generates, together with the volume forces X_i and the temperature field T, the stresses σ_{ij} and deformations ε_{ij} in the state of equilibrium.

We shall specify the properties of an isotropic body by means of a vector function λ with components $\lambda_1(\cdot) = K(\cdot)$, $\lambda_2(\cdot) = G(\cdot)$, or a vector function β, $\beta_1(\cdot) = K^{-1}(\cdot)$, $\beta_2(\cdot) = G^{-1}(\cdot)$. Here K, G denote the volume elasticity and shear moduli. The functions λ, β are assumed to belong to the set Q_+ of vector functions with strictly positive components belonging to the space Q of vector functions piecewise smooth in V, with the norm

$$\|h\| = \operatorname{vrai}\max_{(\cdot)\in V} \max_{i=1,2} |h_i(\cdot)|$$

We shall assume that the properties of the material and deformed state of the body allow the application of the relations of the linear theory of thermoelasticity /5/. Following the variational methods /6/, we can reduce them to an equivalent problem of the minimum of a quadratic functional. We introduce, in the space of square summable in V second rank tensors,

$$\Im = \{t_{ij} \mid t_{ij}(\cdot) = t_{ji}(\cdot), (\cdot) \in V; (t, t) = \int_V t_{ij}t_{ij}\, dV < \infty\} \tag{1.1}$$

the equivalent metric, using the energetic scalar product

$$[\varepsilon, \varepsilon^{(1)}]_\lambda = \int_V \left(\frac{1}{3}\lambda_1\varepsilon_{kk}\varepsilon_{nn}^{(1)} - \lambda_2 e_{ij}e_{ij}^{(1)}\right) dV, \quad \|\varepsilon\|_\lambda^2 = [\varepsilon, \varepsilon]_\lambda \tag{1.2}$$

where e_{ij}, $e_{ij}^{(1)}$ are deviators of the tensors ε_{ij}, $\varepsilon_{ij}^{(1)}$, and repeated indices denote summation.

*Prikl.Matem.Mekhan.,49,3,476-484,1985

We shall use the product (1.2) for the deformation fields. For the stress fields we shall introduce the elastic scalar product generating a metric equivalent to (1.2)

$$[\sigma, \sigma^{(1)}]_\beta = \int_V \left(\frac{1}{3}\beta_1\sigma_{kk}\sigma_{nn}^{(1)} + \beta_2 s_{ij}s_{ij}^{(1)}\right)dV, \quad \|\sigma\|_\beta^2 = [\sigma, \sigma]_\beta \tag{1.3}$$

where $s_{ij}, s_{ij}^{(1)}$ are deviators of the tensors $\sigma_{ij}, \sigma_{ij}^{(1)}$. The stress and deformation tensors are connected by the Duhamel-Neumann law

$$\begin{aligned} &\sigma_{nn} = \lambda_1\varepsilon_{kk}^y, \quad s_{ij} = \lambda_2 e_{ij}^y \\ &\varepsilon_{ij}^y = \varepsilon_{ij} - \varepsilon_{ij}^*, \quad \varepsilon_{ij}^* = \alpha(T - T_0)\delta_{ij} \end{aligned} \tag{1.4}$$

Here α is the linear thermal coefficient of expansion, T_0 is the initial temperature in the undeformed state, and δ_{ij} is the Kronecker delta.

The variational Lagrange and Castigliano principles of the linear problem of thermoelasticity /5/ can be written, using the notation of (1.2), (1.3), in the form

$$\varepsilon : \|\varepsilon - \varepsilon^*\|_\lambda^2 - 2l_H(\varepsilon) \to \min_{\varepsilon\in\Gamma} l_H(\varepsilon) = \int_V \sigma_{ij}^*\varepsilon_{ij}\,dV \tag{1.5}$$

$$\sigma = \sigma^* - \tau : \|\sigma^* - \tau\|_\beta^2 - 2l_T(\tau) \to \min_{\tau\in\Psi} \tag{1.6}$$

$$l_T(\tau) = \int_V \varepsilon_{ij}^*\tau_{ij}\,dV$$

where the space Γ of the kinematically possible deformations is constructed as the complement, in the metric (1.2), to the following space:

$$\begin{aligned} &D = \{\varepsilon \mid \exists u_i \in C^2(V): \ \varepsilon_{ij} = {}^1/_2(u_{i,j} + u_{j,i}) \ u_i(\cdot) = 0, \\ &(\cdot) \in A_1\}: \Gamma = \bar{D} \end{aligned} \tag{1.7}$$

The space Ψ of statically possible stresses is constructed as the complement, in the metric (1.3), to the space

$$\begin{aligned} &M = \{\sigma \mid \sigma_{ij} \in C^1(V), \ \sigma_{ji,j} = 0 \text{ on } V, \ \sigma_{ji}n_j = 0 \text{ on } A_2\} \\ &\Psi = \bar{M} \end{aligned} \tag{1.8}$$

Here $C^k(V)$ is a space of functions k-times continuously differentiable on V (the index after the comma indicates differentiation with respect to the corresponding coordinate), and n_j is the vector normal to the surface. The symbol σ_{ij}^* in (1.5), (1.6) denotes an arbitrary particular solution satisfying, almost everywhere on V, the equations of equilibrium in the stresses and the force boundary conditions on A_2.

We further assume that the tensors ε^*, σ^* are square summable on V. The problem (1.5), (1.6) represent special cases of the problems on a minimum of quadratic functionals /7/ whose conditions of minimum

$$\varepsilon \in \Gamma: [\varepsilon - \varepsilon^*, \varepsilon^{(1)}]_\lambda = l_H(\varepsilon^{(1)}), \ \forall \varepsilon^{(1)} \in \Gamma \tag{1.9}$$

$$\tau \in \Psi: [\sigma^* - \tau, \tau^{(1)}]_\beta = -l_T(\tau^{(1)}), \ \forall \tau^{(1)} \in \Psi \tag{1.10}$$

uniquely define the deformations ε and stresses $\sigma = \sigma^* - \tau$.

Let us now denote by Λ_ε the set of distributions of elastic moduli $\lambda \in Q_+$ for which one and the same deformation field ε is realized in the state of equilibrium under the action of the given load and the temperature field. Correspondingly, B_σ is the set of distributions of elastic compliances $\beta \in Q_+$, realizing the distribution of stresses σ.

We shall show that $\Lambda_\varepsilon, B_\sigma$ have an affine structure. We shall select two designs realizing the same distribution of deformations $\lambda' \in \Lambda_\varepsilon, \lambda'' \in \Lambda_\varepsilon$, and consider the admissible design $\lambda \in Q_+$, lying on the straight line connecting λ', λ''. Writing condition (1.9) for each design, multiplying the first relation by p and combining with the second relation multiplied by $(1-p)$, we again obtain, by virtue of the linear dependence of the scalar product (1.2) on λ, conditions (1.9) where $\lambda = (p\lambda' + (1-p)\lambda'') \in \Lambda_\varepsilon$. Carrying out a similar analysis of relations (1.10), we can conclude that the sets $\Lambda_\varepsilon, B_\sigma$ contain, together with any two of their points, a part of the straight line connecting these points and lying in Q_+, i.e. $\Lambda_\varepsilon, B_\sigma$ have an affine structure

$$\begin{aligned} &\lambda' \in \Lambda_\varepsilon, \ \lambda'' \in \Lambda_\varepsilon \Rightarrow (p\lambda' + (1-p)\lambda'') \in \Lambda_\varepsilon \\ &\forall p \in R: \ p\lambda' + (1-p)\lambda'' \in Q_+; \ \beta' \in B_\sigma, \ \beta'' \in B_\sigma \Rightarrow \\ &\quad (p\beta' + (1-p)\beta'') \in B_\sigma, \ \forall p \in R: \ p\beta' + (1-p)\beta'' \in Q_+ \end{aligned} \tag{1.11}$$

2. We shall investigate the relations connecting the sets of designs Λ_ε, realizing the same deformation field, with the sets of designs B_σ, realizing the same stress field, and obtain the explicit expressions for the sets $\Lambda_\varepsilon, B_\sigma$.

We shall introduce the space Υ_σ of stresses corresponding to the deformations form Υ, in accordance with Hooke's law, as the closure inthe metric (1.3) of the following space:

$$D_\sigma = \{\sigma \mid \exists \varepsilon \in D: \sigma_{kk} = \lambda_1 \varepsilon_{nn}, \ s_{ij} = \lambda_2 e_{ij}\} \quad (2.1)$$
$$\Upsilon_\sigma = \bar{D}_\sigma$$

We shall also introduce the space Ψ_ε of deformations corresponding to the stresses from Ψ in accordance with Hooke's law, as a closure in the metric (1.2) of the following space:

$$M_\varepsilon = \{\varepsilon \mid \exists \sigma \in M: \varepsilon_{kk} = \beta_1 \sigma_{nn}, \ e_{ij} = \beta_2 s_{ij}\} \quad (2.2)$$

where the spaces D, M are given by (1.7), (1.8).

In what follows, we shall make use of the fact that the spaces introduced are connected by the relations

$$\Theta = \Psi \oplus \Upsilon_\sigma, \ \Theta = \Upsilon \oplus \Psi_\varepsilon. \quad (2.3)$$

The first property was proved in /6/. The second property is equivalent to the following two assertions:

$$[\varepsilon', \varepsilon'']_\lambda = 0, \ \forall \varepsilon' \in \Psi_\varepsilon, \ \forall \varepsilon'' \in \Upsilon \quad (2.4)$$
$$\forall \xi \in \Theta, \ \exists \varepsilon' \in \Psi_\varepsilon, \ \exists \varepsilon'' \in \Upsilon: \ \xi = \varepsilon' + \varepsilon'' \quad (2.5)$$

Property (2.4) is proved by passing to the limit in the relation describing the orthogonality of the spaces D, M_ε, obtained by integration using Gauss's formula. To prove relation (2.5), we shall consider the problem of the equilibrium of a body acted upon by the distortion ξ /5/ in the absence of an external load. Using condition (1.5), we obtain the problem of the minimum of a quadratic functional with the following linear functional bounded in Υ:

$$\varepsilon'': \|\varepsilon'' - \xi\|_\lambda^2 \to \min_{\varepsilon'' \in \Upsilon}, \quad 2 |[\xi, \varepsilon]_\lambda| \leqslant \alpha \|\varepsilon\|_\lambda$$
$$\alpha = 2 (\xi, \xi)^{1/2} \max_{(\cdot) \in V} \max_{i=1,2} \lambda_i (\cdot)$$

Solutions of such problems exist and are unique /7/. The stresses arising in this case are selfbalancing; therefore the corresponding purely elastic parts of the deformations $\varepsilon' = (\varepsilon'' - \xi)$ belong to Ψ_ε. The deformations ε'' belong to Υ by virtue of thier construction. This proves property (2.5), and hence (2.3).

We shall find a relation connecting the sets $\Lambda_\varepsilon, B_\sigma$ using the dependence of the stresses and deformations on the design of the distribution of elastic moduli. We consider, together with the design λ (or β) from Q_+ and the corresponding deformations $\varepsilon = \varepsilon(\lambda)$ and stresses $\sigma = \sigma(\beta)$, another design $\lambda' = \lambda + \mu$ (or $\beta' = \beta + \gamma$). The corresponding stresses σ' and deformations ε' are found using the perturbation method, in the form of the series /8/

$$\sigma' = \sigma + \sum_{n=1}^{\infty} (-\Gamma_\gamma)^n \sigma, \quad \varepsilon' = \varepsilon + \sum_{n=1}^{\infty} (-\Delta_\lambda)^n (\varepsilon - \varepsilon^*) \quad (2.6)$$

where the bilinear operators $\Gamma: Q \times \Theta \to \Psi, \ \Delta: Q \times \Theta \to \Upsilon$ are defined by the conditions

$$\gamma \in Q, \ \sigma \in \Theta, \ (\Gamma_\gamma \sigma) \in \Psi: \ [\Gamma_\gamma \sigma, \tau]_\beta = [\sigma, \tau]_\gamma, \ \forall \tau \in \Psi \quad (2.7)$$
$$\mu \in Q, \ \varepsilon \in \Theta, \ (\Delta_\mu \varepsilon) \in \Upsilon: [\Delta_\mu \varepsilon, \varepsilon']_\lambda = [\varepsilon, \varepsilon']_\mu, \ \forall \varepsilon' \in \Upsilon \quad (2.8)$$

The quantities $[.]_\gamma, [.]_\mu$ are found using the formulas (1.2), (1.3) where β is replaced by γ, λ by μ. Following /8/, we can show that series (2.6) converge on the energetic norm in the regions containing, respectively, the spheres

$$\|\beta' - \beta\| < r_1, \quad r_1 = \operatorname{vrai} \min_{(\cdot) \in V} \min_{i=1,2} \beta_i \ \|\lambda' - \lambda\| < r_2 \quad (2.9)$$
$$r_2 = \operatorname{vrai} \min_{(\cdot) \in V} \min_{i=1,2} \lambda_i (\cdot)$$

We shall assume that the distribution of the moduli of λ undergoes an infinitesimal change $\delta\lambda$, when the corresponding change in the compliances is determined after the differentiation

$$\beta_i^{-1} (\cdot) \cdot \delta\beta_i (\cdot) = -\lambda_i^{-1} (\cdot) \cdot \delta\lambda_i (\cdot), \ (\cdot) \in V, \ i = 1, 2 \quad (2.10)$$

When varying the stresses $\delta\sigma$ and deformations $\delta\varepsilon$ in series (2.6), we shall restrict ourselves to quantities of the first order of smallness

$$\delta\sigma \in \Psi: \ [\delta\sigma, \sigma']_\beta = -[\sigma, \sigma']_{\delta\beta}, \ \forall \sigma' \in \Psi$$
$$\delta\varepsilon \in \Upsilon: \ [\delta\varepsilon, \varepsilon']_\lambda = -[\varepsilon - \varepsilon^*, \varepsilon']_{\delta\lambda}, \ \forall \varepsilon' \in \Upsilon$$

Let the tensors ε', σ' be connected by Hooke's law (1.4). Then, using (2.10) (1.2), (1.3), we can obtain the property

$$[\sigma, \sigma']_{\delta\beta} = -[\varepsilon - \varepsilon^*, \varepsilon']_{\delta\lambda}$$

which enables us to write the conditions of invariance of the deformations and stresses to a small variation in the design, in the form

$$\delta\sigma = 0 \Leftrightarrow [q, \varepsilon']_\lambda = 0, \ \forall \varepsilon' \in \Psi_\varepsilon \qquad (2.11)$$
$$\delta\varepsilon = 0 \Leftrightarrow [q, \varepsilon']_\lambda = 0, \ \forall \varepsilon' \in \Upsilon$$

where the tensor q is given by the relations

$$q_{kk} = \delta\lambda_1 \cdot \lambda_1^{-1} (\varepsilon_{kk} - \varepsilon_{kk}^*), q_{ij} - {}^1/_3 q_{kk}\, \delta_{ij} = \delta\lambda_2 \cdot \lambda_2^{-1} (\varepsilon_{ij} - \varepsilon_{ij}^* - {}^1/_3 \varepsilon_{kk} \delta_{ij}) \qquad (2.12)$$

Property (2.3) consisting of the fact that the space Υ is an orthogonal complement Ψ_ε to the space Э, yields the relation connecting the first and second condition of (2.11). First, according to (2.4) both these conditions cannot be satisfied simultaneously when $q \neq 0$, i.e., there is no direction in which the elastic moduli $\delta\lambda$ undergo a small change without a change in the deformations and stresses. Secondly, according to (2.5) any small change in the design $\delta\lambda$ can be represented uniquely in the form of a sum $\delta\lambda' + \delta\lambda''$ so that the change in the design $\delta\lambda'$ will not affect the stresses and a change in $\delta\lambda''$ will not affect the deformations.

We can carry out a suitable analysis based on the first relation of (2.3) for small elastic compliances $\delta\beta'$ not affecting the stresses, and for the changes in $\delta\beta''$ not affecting the deformations.

Let us compare the set Λ_ε of distributions of the elastic moduli realizing the same deformation field ε, with the set B_ε of distributions of the corresponding elastic compliances. Let us also compare the set B_σ of distributions of elastic compliances realizing a single stress field σ, with the set Λ_σ of the corresponding distributions of the elastic moduli. Then the results obtained here have the following geometrical interpretation. The family of affine sets Λ_ε covers, together with the family of sets Λ_σ, the whole set of possible designs Q_+. Every set Λ_ε intersects the set Λ_σ at one point at the most, and the spaces constructed, tangent to Λ_ε and Λ_σ at this point complement each other. The families of affine sets B_σ and B_ε have analogous properties.

The sets $\Lambda_\varepsilon, B_\sigma$ can be constructed in terms of one of their representative elements $\lambda \in \Lambda_\varepsilon$, $\beta \in B_\sigma$ as follows:

$$\Lambda_\varepsilon = \{(\lambda - \mu) \in Q_+ : [\varepsilon - \varepsilon^*, \varepsilon']_\mu = 0, \ \forall \varepsilon' \in \Upsilon\} \qquad (2.13)$$

$$B_\sigma = \{(\beta + \gamma) \in Q_+ : [\sigma, \sigma']_\gamma = 0, \ \forall \sigma' \in \Psi\} \qquad (2.14)$$

To prove (2.13) we assume that the increment of the design μ satisfies the condition

$$\mu \in Q : [\varepsilon - \varepsilon^*, \varepsilon']_\mu = 0, \ \forall \varepsilon' \in \Upsilon \qquad (2.15)$$

Let us choose a positive number t_1, so that the design $\lambda' = \lambda + t_1\mu$ belongs to the sphere (2.9) of convergence of the series (2.6) for ε'. According to condition (2.15), the element $\Delta_\mu (\varepsilon - \varepsilon^*)$, found from (2.8) and the remaining elements $(-\Delta_\mu)^n (\varepsilon - \varepsilon^*)$, are all zero. This means that $(\lambda + t_1\mu) \in \Lambda_\varepsilon$. But then, according to property (1.11) of the affinity of the set Λ_ε, the whole part of the straight line $\lambda + t\mu$, lying within the admissible set Q_+ also lies in Λ_ε. In other words, the designs constructed according to (2.13) indeed belong to the set Λ_ε. It remains to show that construction (2.13) exhausts all elements of the set Λ_ε. Let us assume the opposite, i.e. that

$$\exists\, (\lambda + \mu) \in \Lambda_\varepsilon : \exists \varepsilon' \in \Upsilon : [\varepsilon - \varepsilon^*, \varepsilon']_\mu \neq 0 \qquad (2.16)$$

The property of the affinity of Λ_ε (1.11) implies that the whole part of the straight line $(\lambda + t_\mu)$ belongs to Λ_ε. Let us choose, on this part, a rectilinear segment $t \in [0, t_1]$, such that the power series in t

$$[\varepsilon(\lambda + t\mu), \varepsilon']_\lambda = [\varepsilon, \varepsilon']_\lambda + \sum_{n=1}^{\infty} c_n t^n$$
$$c_n = [(-\Delta_h)^n (\varepsilon - \varepsilon^*), \varepsilon']_\lambda$$

converges. Since the above series represents an expansion of a zero function defined on $[0, t_1]$, it follows from the well-known property /9/ of the uniqueness of a power series that all the coefficients c_n must vanish, and this contradicts, at $n = 1$ assumption (2.16).

Relation (2.13) determining the explicit form of the affine subset Λ_ε, is thus completely proved. Relation (2.14) determining the affine structure of the set B_σ, is proved in the same manner.

3. In designing certain highly accurate instruments (e.g. radio telescopes), the requirement of strength is accompanied by the demand that the displacement (deformation) field be of prescribed form. In this connection we pose the following model problem of optimal design. We seek, for a thermoelastic body discussed in Sect.1, a distribution of elastic moduli such that a given deformation field is realized in the state of equilibrium at the minimum stress level. Below we shall show that the problem formulated here refers to the well-known class of problems of convex programming /10/.

We shall measure the stress level corresponding to the design λ, using the quantity

$$p(\lambda)=\|\sigma(\lambda)\|,\ \|\sigma\|=\max_{(\cdot)\in V}\Gamma(\sigma(\cdot)) \tag{3.1}$$

$$\Gamma(\sigma)=(c_1\sigma_{kk}\sigma_{nn}+c_2s_{ij}s_{ij})^{1/2}$$

where c_1, c_2 are given positive numbers and s is the deviator of the tensor σ. The property of positive definiteness and uniformity of the norm $\|\sigma\|$ is obvious, and the triangle inequality for it follows from the property $\Gamma(\sigma'+\sigma'')\leqslant\Gamma(\sigma')+\Gamma(\sigma'')$ and the properties of the max operation over the region V.

Using Hooke's law (1.4) and property (1.11) of the affinity of the set Λ_ε, we obtain the affinity of the set of the corresponding stresses

$$\lambda'\in\Lambda_\varepsilon,\ \lambda''\in\Lambda_\varepsilon,\ t\lambda'+(1-t)\lambda''\in Q_+\Rightarrow\sigma(t\lambda'+(1-t)\lambda'')=t\sigma(\lambda')+(1-t)\sigma(\lambda'') \tag{3.2}$$

The relations (3.1), (3.2) enable us to construct the following sequence of inequalities proving the convexity of the functional $p(\lambda)$ on Q_+:

$$\begin{aligned}&p(t\lambda'+(1-t)\lambda'')=\|\sigma(t\lambda'+(1-t)\lambda'')\|=\\&\|t\sigma(\lambda')+(1-t)\sigma(\lambda'')\|\leqslant t\|\sigma(\lambda')\|+(1-t)\|\sigma(\lambda'')\|=\\&tp(\lambda')+(1-t)p(\lambda''),\ t>0,\ (1-t)>0\end{aligned}$$

Therefore the problem of optimal design

$$p(\lambda)\to\min_\lambda,\ \lambda\in\Lambda_\varepsilon \tag{3.3}$$

is a problem of minimizing the convex functional $p(\lambda)$ on the convex set Λ_ε, i.e. a problem of convex programming /10/. We note that the problem of designing a body of optimal rigidity for the given stress field is solved inthe same manner, and also represents a problem of convex programming.

4. We shall consider, as an example, the problem of determining the optimal distribution of Young's modulus $E(x)$ over the length of a thermoelastic beam, from the condition of the lowest maximum value of the stress modulus σ for the given distribution of the flexure $w(x)$. A beam, rigidly clamped at both ends $x=0, x=l$, is in an equilibrium state under the action of a distributed transverse load $q(x)$ and temperature field $T(x,z)$. Using the hypothesis of plane sections and assuming that the temperature field is compatible with the conditions of pure bending, we obtain the equations of equilibrium in the form /11/

$$(Ef)''=q;\quad f=Iw''-m,\quad I=\int_A z^2\,dA,\quad m=\int_A\alpha Tz\,dA \tag{4.1}$$

Here A is the area, generally speaking, of a transverse cross-section of the beam.

Integrating Eq.(4.1), we obtain the law of distribution of Young's modulus

$$E(x)=\frac{c_1x+c_2+\varphi(x)}{f(x)},\quad \varphi(x)=\int_0^x dt\int_0^t q(s)\,ds \tag{4.2}$$

for the values of x for which $f(x)\neq0$. In what follows, we shall assume that the given function $f(x)$ has a finite number of roots on $[0,l]$. The undefined constants c_1 and c_2 must satisfy the condition $E(x)>0$, i.e.

$$\operatorname{sign}(c_1x+c_2+\varphi(x))=\operatorname{sign}f(x) \tag{4.3}$$

The property of affinity of the set of distributions $E(x)$ constructed according to (4.2), (4.3), can be confirmed directly.

The possibility of satisfying condition (4.3) depends on the specified distributions of w and m. If the flexure $w(x)$ is realized for some distribution $E(x)>0$, then condition (4.3) will hold for at least one pair of values of c_1, c_2. If, on the other hand, condition (4.3) admits of an arbitrariness in the choice of the parameters c_1, c_2, then it can be used to reduce the stress level

$$\|\sigma\|=\max_{x\in[0,l]}\max_{(y,z)\in A}|\sigma(x,y,z)|\to\min_{c_1,c_2} \tag{4.4}$$

$$(\sigma=(c_1x+c_2+\varphi(x))\Phi\quad\text{for}\quad f\neq0;\ \sigma=0\quad\text{for}\quad f=0;\ \Phi=(zw''-\alpha T)/f)$$

The convex nature of problem (4.3), (4.4) can be confirmed by direct substitution. Below we give an analytic solution of this problem for the following initial data:

$$q(x) = q = \text{const}; \quad T(x, z) = \theta(\hat{x}) \cdot z; \quad W(x) = W_0 = \text{const} \tag{4.5}$$

where $W(x)$ is the moment of the reaction of the beam at the cross-section x. When the distributed external load is constant, the bending moment M has a parabolic distribution and the initial data for the function $f = ME^{-1}$ must therefore correspond to one of the following cases:

$$f(x) > 0,\ x \in \,]0, x_0[;\ f(x) < 0,\ x \in \,]x_0, l[;\ x_0 \in \,]0, l[\tag{4.6}$$
$$f(x) < 0,\ x \in \,]0, x_0[;\ f(x) > 0,\ x \in \,]x_0, l[;\ x_0 \in \,]0, l[\tag{4.7}$$
$$f(x) \geqslant 0,\ x \in [0, l] \tag{4.8}$$
$$f(x) \leqslant 0,\ x \in [0, l] \tag{4.9}$$
$$f(x) > 0,\ x \in \,]0, x_1[;\ f(x) < 0,\ x \in \,]x_1, x_2[,\ f(x) > 0,\ x \in \,]x_2, l[;\quad 0 < x_1 < x_2 < l \tag{4.10}$$
$$f(x) < 0,\ x \in [0, l],\ x \neq x_0,\ f(x_0) = 0,\ x_0 \in [0, l] \tag{4.11}$$

Substituting (4.5) into (4.4) we obtain the problem

$$\|\sigma\| = W_0^{-1} \max_{x \in [0, l]} |r(c_1, c_2, x)| \to \min_{c_1, c_2}$$
$$r = c_1 x + c_2 + {}^1/_2 q x^2; \ \operatorname{sign} r(c_1, c_2, x) = \operatorname{sign} f(x),\ x \in [0, l] \tag{4.12}$$

After solving this problem we shall find the undefined constants c_1, c_2 appearing in expression (4.2) for the optimal distribution of Young's modulus in the beam

$$E(x) = (Iw'' - m)^{-1} r,\ r = c_1 x + c_2 + {}^1/_2 q x^2 \tag{4.13}$$

As we shall see below, the method of solving (4.12) will depend on which of the conditions (4.6) – (4.11) is satisfied by the function f. In case (4.6), problem (4.12) can be reduced, using the substitution $c_1 = -t x_0^{-1} - {}^1/_2 x_0 q,\ c_2 = t$, to the form

$$\|\sigma\| = W_0^{-1} \max_{x \in [0, l]} |S(x, t)| \to \min_{t \geqslant {}^1/_2 q x_0 l}$$
$$S(x, t) = (x - x_0)({}^1/_2 q x - t x_0^{-1})$$

We note that if we fix $x \in \,]0, x_0[$, then $S(x, t) > 0$ will increase with t. If we fix $x \in \,]x_0, l[$, then $S(x, t) < 0$ will decrease with t. Therefore a global optimum is attained at $t = {}^1/_2 q x_0 l$, and the optimal distribution of Young's modulus (4.13) has the form (4.13) when $r = {}^1/_2 q (x - x_0)(x - l)$.

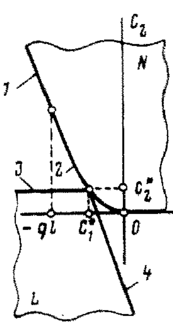
Case (4.7) reduces to the case already discussed, by changing the direction and the origin of the Ox axis.

In case (4.8) the parameters c_1, c_2 are chosen so that the graph of the parabola $r = r(x)$ lies above the segment $[0, l]$ of the Ox axis, or, if the points of intersection of the parabola with the axis lie to the left of the segment $[0, l]$, of if these points lie to the right of the segment $[0, l]$. The requirements listed above are satisfied on the set N formed by combining the following three sets:

$$N = \left\{c_2 > \frac{1}{2q} c_1^2\right\} \cup \left\{\frac{1}{2q} c_1^2 > c_2 > 0;\ c_1 > 0\right\} \cup \left\{\frac{c_1^2}{2q} > c_2 > -c_1 l - \frac{1}{2} q l^2;\ c_1 < -ql\right\}$$

This set (see the figure) has a lower bound fixed by a line composed of a straight line 1: $c_2 = -c_1 l - {}^1/_2 q l^2$ on the segment $c_1 \in \,]-\infty, -ql[$, the parabola 2: $c_2 = {}^1/_2 q c_1^2$ on the segment $c_1 \in \,]-ql, 0[$ and the Oc_1 axis on the half-line $c_1 \in [0, \infty[$. The largest stress modulus occurs at one of the beam ends

$$\|\sigma\| = W_0^{-1} \max_{x \in [0, l]} |r(c_1, c_2, x)| = W_0^{-1} \max\{|r(c_1, c_2, 0)|, |r(c_1, c_2, l)|\} = W_0^{-1} \max\{c_2, c_2 - (c_1 - {}^1/_2 q l) l\}$$



The figure depicts the set L of parameters c_1, c_2, for which $\|\sigma\| \leqslant {}^1/_8 W_0^{-1} q l^2$. The set L has an upper bound fixed by a broken line composed of the straight line 3 $c_2 = {}^1/_8 q l^2$ for $c_1 \in \,]-\infty, -{}^1/_2 q l[$ and the straight line 4: $c_2 = {}^1/_8 q l^2 - (c_1 + {}^1/_2 q l) l$, $c_1 \in \,]-{}^1/_2 q l, \infty[$. We see that the sets N and L have one common point $c_1^* = -{}^1/_2 q l$, $c_2^* = {}^1/_8 q l^2$, which also defines the optimal distribution of Young's modulus for case (4.8) in the form (4.13) with $r = {}^1/_2 q (x - {}^1/_2 l)^2$.

Considering the case (4.9), we can show that every parabola $r = c_1 x + c_2 + {}^1/_2 q x^2$ which takes non-positive values on the segment $[0, l]$, has a graph situated not higher than the graph of the parabola $\frac{1}{2}qx(x-l)$ passing through the points $(0, 0)$; $(l, 0)$. Therefore the values $c_1^* = -\frac{1}{2}ql$, $c_2^* = 0$ are optimal and the law of distribution of Young's modulus for case (4.9) corresponds to (4.13) with $r = \frac{1}{2}qx(x-l)$.

Finally, the conditions (4.10), (4.11) uniquely define the values of the parameters c_1, c_2, i.e. in everyone of these cases the process of specifying the distributions of the flexures does not leave any arbitrariness for reducing the stress level. The distributions of Young's modulus have the form (4.13) for $r = \frac{1}{2}q(x-x_1)(x-x_2)$, $r = \frac{1}{2}q(x-x_0)^2$ respectively.

The results obtained in Sect. 1 and 2 can be illustrated by the example in question. Here we must remember that the bending moment M represents the generalized stress, and the variation in curvature $\varkappa$ is the generalized deformation.

Let us introduce the energetic scalar product analogous to (1.3)

$$[M, M^{(1)}]_\beta = \int_0^l MM^{(1)}\beta I^{-1}\,dx, \quad \beta = E^{-1} \tag{4.14}$$

Here the space Ψ (1.8) of selfbalanced stresses has a corresponding space of bending moments satisfying the following homogeneous equations of equilibrium $M'' = 0$:

$$\Psi = \{M \mid M = c_1x + c_2,\ c_1 \in R,\ c_2 \in R\} \tag{4.15}$$

The Duhamel-Neumann relations (1.4) have the form

$$M = \beta^{-1}I(\varkappa - \varkappa^*),\ \varkappa^* = mI^{-1},\ \varkappa = w'' \tag{4.16}$$

Using this to obtain the expression for w'', we carry out the integration and substitute the boundary conditions $w(l) = w'(l) = 0$. This yields the following set of equations:

$$[M, x]_\beta = a_1,\quad [M, 1]_\beta = a_2,\quad M = c_1x + c_2 + \varphi,\quad a_k = -\int_0^l x^{2-k}mI^{-1}\,dx \tag{4.17}$$

Let us consider, together with the design of the distribution of the compliance β and the corresponding bending moment M, another design $(\beta+\gamma)$ with bending moment $M + \Delta M$. Using system (4.17) we obtain

$$[\Delta M, x]_{\beta+\gamma} = [M, x]_\gamma,\quad [\Delta M, 1]_{\beta+\gamma} = [M, 1]_\gamma$$
$$(\Delta M = x\Delta c_1 + \Delta c_2) \tag{4.18}$$

When the variation in the design γ is specified, the above system yields uniquely Δc_1, Δc_2, since its discriminant is, by virtue of the Cauchy-Schwartz inequality, different from zero. From system (4.18) it follows that in order to have the same bending moments in the designs β and $\beta+\gamma$ of the beam, it is necessary and sufficient that

$$[M, M^{(1)}]_\gamma = 0,\ \forall M^{(1)} \in \Psi \Rightarrow \Delta M = 0 \tag{4.19}$$

Condition (4.19) is identical apart from the notation, with (2.14), reveals the affine structure of the designs with the same stress field, and can be used inthe problem of optimizing the beam flexure for a given state of stress.

Let the designs of the distribution of the compliance β undergo an infinitesimal change $\delta\beta$. Then the change in curvature calculated from (4.16) will be

$$\delta w'' = I^{-1}\delta(M\beta)$$

Since $\delta M \in \Psi$, and by virtue of the kinematic boundary conditions $\delta w'' = 0 \Leftrightarrow \delta w = 0$, the condition of invariance of the flexure to a small variation in the design has the form

$$\delta w = 0 \Leftrightarrow g \in \Psi;\ g = \delta\beta\cdot\beta^{-1}M \tag{4.20}$$

At the same time, from (4.19) it follows that the necessary and sufficient condition for invariance of the stresses is, that the relative change in compliance g be orthogonal to Ψ, i.e.

$$\delta M = 0 \Leftrightarrow [g, M^{(1)}]_\beta = 0,\ \forall M^{(1)} \in \Psi \tag{4.21}$$

The relations (4.20), (4.21) correspond to the result of Sect. 2 (see (2.11)) according to which any small change in the design $\delta\beta$ can be represented in the form of a sum of the change $\delta\beta'$, leaving the displacements unchanged, and $\delta\beta''$, leaving the stresses unchanged. The result can be utilized in new problems of optimal design, taking into account simultaneously the reduction in the level of stresses and the deformations. For example, it follows from it that a small change inthe design can be found, for which the level of stresses $\|\sigma(\beta)\|$ and deformations $\|\varepsilon(\beta)\|$ are both reduced. This is connected with the fact that the gradients of the functionals $\|\sigma(\beta)\|$ and $\|\varepsilon(\beta)\|$ lie in mutually complementary subspaces.

REFERENCES

1. LUR'E K.A., Optimal Control in Problems of Mathematical Physics. Moscow, Nauka, 1975.
2. LAVROV N.A., LUR'E K.A. and CHERKAEV A.V., Inhomogeneous rod of extremal torsional rigidity. Izv. Akad. Nauk SSSR, MTT, No.6, 1980.
3. PETROVA I.S. and RIKARDS R.B., Optimization of a rod with variable modulus of elasticity. Mekhanika polymerov, No.2, 1974.
4. RAMMERSTORFER F., On the optimal distribution of the Young's modulus of a vibrating, prestressed beam. J. Sound and Vibr., Vol.37, No.1, 1974.
5. NOWACKI W., Thermoelasticity. Reading, Mass., Addison-Wesley Publ. 1962.
6. MIKHLIN S.G., Variational Methods in Mathematical Physics. Moscow, Nauka, 1970.
7. MIKHLIN S.G., The Problem of the Minimum of a Quadratic Functional. Moscow-Leningrad, Gostekhizdat, 1952.
8. KUNTASHEV P.A. and NEMIROVSKII YU.V., On the solution, in terms of stresses, of the problem of the thermoelasticity of inhomogeneous bodies using the perturbation method. PMM Vol.49, No.2, 1985.
9. RUDIN W., Principles of Mathematical Analysis. N.Y., London, McGraw-Hill, 1976.
10. PSHENICHNYI B.N., Convex Analysis and Extremal Problems. Moscow, Nauka, 1980.
11. BOLEY B.A. and WEINER J.H., Theory of Thermal Stresses. N.Y. Wiley, 1960.

Translated by L.K.

PMM U.S.S.R., Vol.49, No.3, pp. 374-379, 1985 0021-8928/85 $10.00+0.00
Printed in Great Britain Pergamon Journals Ltd.

THE PLANE PROBLEM OF ELECTROELASTICITY FOR A PIEZOELECTRIC LAYER WITH A PERIODIC SYSTEM OF ELECTRODES AT THE SURFACES*

V.A. KOKUNOV B.A. KUDRYAVTSEV AND N.A. SENIK

Static electroelasticity equations are used to study the stress state and the electric-field distribution in a piezoelectric layer at whose surface a periodic system of infinitely thin electrodes is situated. It is assumed that the layer material is piezoelectric belonging to the $6mm$ symmetry class, and the axis of symmetry is perpendicular to the middle surface of the layer. The mechanical displacements and electric potential are determined, taking the periodicity of the electrode system into account, in the form of trigonometric series, and the electrical and mechanical boundary conditions at the layer surfaces lead to the dual series equations whose solution yields the expression for the electric charge distribution density on each electrode. Formulas are given for determing the electric potential at the layer surfaces between the electrodes, and the mechanical stresses near the electrode edge. It is shown that the normal stresses at the layer surface have a singularity at the electrode edge /1/ whose presence may lead to the appearance of microcracks within this zone.

1. We shall consider the plane deformation of a piezoelectric layer $|z|<h,\ |x|<\infty$ caused by the action of the electric potential difference on the periodic system of electrodes, with the electric potentials V_0 and $-V_0$ on the upper face $z=h$ and lower face $z=-h$ of the layer (Fig.1). In the case of a piezoelectric material of class $6mm$, whose axis of symmetry coincides with the z-axis, the components of the stresses and electric induction are given by the formulas

$$\sigma_{xx}=c_{11}\frac{\partial u}{\partial x}+c_{13}\frac{\partial w}{\partial z}+e_{31}\frac{\partial\varphi}{\partial z},\quad \sigma_{zz}=c_{13}\frac{\partial u}{\partial x}+c_{33}\frac{\partial w}{\partial z}+e_{33}\frac{\partial\varphi}{\partial z} \tag{1.1}$$

$$\sigma_{zx}=c_{44}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+e_{15}\frac{\partial\varphi}{\partial x}$$

$$D_x=e_{15}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)-\varepsilon_{11}{}^S\frac{\partial\varphi}{\partial x},\quad D_z=e_{31}\frac{\partial u}{\partial x}+e_{33}\frac{\partial w}{\partial z}-\varepsilon_{33}{}^S\frac{\partial\varphi}{\partial z} \tag{1.2}$$

Here $c_{11}, c_{13}, c_{33}, c_{44}$ are the moduli of elasticity, e_{31}, e_{33}, e_{15} are the piezoelectric moduli, $\varepsilon_{11}{}^S$, $\varepsilon_{33}{}^S$ are the dielectric constants, u, w are the components of the displacement vector in the direction of the x and z axes respectively, and φ is the electric potential.

The mechanical displacements u, w and electric potential are found from the system of

*Prikl.Matem.Mekhan., 49, 3, 485-491, 1985.